Left-over of last Friday:

Recall: Existence & Uniqueness theorem for linear ODE

For the IVP $y' + p(t)y = g(t)$, $y(t_0) = y_0$

satisfying (1) $p(t)$, $g(t)$ continuous over (a,b)

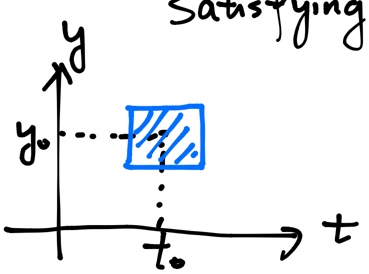
(2) t_0 contained in (a,b)

Then the IVP has a unique solution on (a,b)

Nonlinear Version:

For any nonlinear IVP

$$y' = f(t,y), \quad y(t_0) = y_0$$



satisfying (1) $f(t,y)$ as a two-variable function, is continuous

NEAR (t_0, y_0), i.e., one can find small ε_1, η_1

s.t. $f(t,y)$ continuous $\boxed{(t_0-\varepsilon_1, t_0+\varepsilon_1) \times (y_0-\eta_1, y_0+\eta_1)}$

(2) $\frac{\partial f}{\partial y}(t,y)$ as two variable function, is continuous

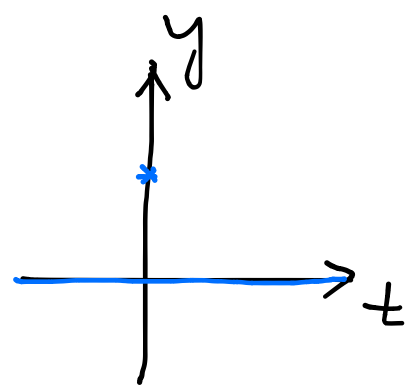
NEAR (t_0, y_0)

then there exists a unique solution NEAR $t = t_0$, i.e. one can find small $\varepsilon > 0$ s.t. on the interval $(t_0 - \varepsilon, t_0 + \varepsilon)$, there's a unique solution.

Remarks:

* The theorem is not as strong as that of linear ODEs. It only concludes the local existence. There's no way to determine how large can ε be.
* Nevertheless, this theorem tells if an IVP is pathological formulated.

Examples:



① $y' = y^{1/3}$, $y(0) = 1$

Rmk: Don't forget to check f_y.

$f(t,y) = y^{\frac{1}{3}}$ continuous everywhere.

$\frac{\partial f}{\partial y}(t,y) = \frac{1}{3} y^{-\frac{2}{3}} = \frac{1}{3y^{2/3}}$ continuous where $y \neq 0$.

$y(0) = 1 \Rightarrow (t_0, y_0) = (0, 1). \Rightarrow$ Existence & Uniqueness near $t=0$

② $y' = y^{1/3}$, $y(1) = 0$

$\frac{\partial f}{\partial y}$ is not continuous near $(1, 0)$

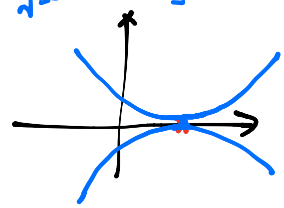
We cannot conclude existence or uniqueness of the sol'n to the IVP.

$\frac{dy}{y^{\frac{1}{3}}} = dx \Rightarrow y^{-\frac{1}{3}} dy = dx \Rightarrow \frac{1}{-\frac{1}{3}+1} y^{-\frac{1}{3}+1} = x + C \Rightarrow$

$$\Rightarrow \frac{3}{2} y^{\frac{2}{3}} = x + C. \Rightarrow 0 = 1 + C \Rightarrow C = -1.$$

$$\frac{3}{2} y^{\frac{2}{3}} = x - 1 \Rightarrow y^2 = \left[\frac{2}{3}(x-1)\right]^3 \Rightarrow y = \pm\sqrt{\left[\frac{2}{3}(x-1)\right]^3}$$

$y(1)=0 \Rightarrow$ Cannot determine $+$ or $-$.



Example: $y' = (1-x^2-y^2)^{1/2}$, $y(x_0) = y_0$

Find the region in x-y plane for (x_0, y_0) such that the above IVP is reasonably formulated.

$f(x,y) = (1-x^2-y^2)^{\frac{1}{2}}$ is continuous when $1-x^2-y^2 \geq 0$

$\frac{\partial f}{\partial y}(x,y) = \frac{-2y}{2\sqrt{1-x^2-y^2}} = -\frac{y}{\sqrt{1-x^2-y^2}}$ is continuous when $1-x^2-y^2 > 0$

The region we're looking for is $\{(x,y): x^2+y^2 < 1\}$

open unit disk.

Qualitative Method for first order autonomous ODEs

Autonomous ODE $y' = f(y)$

RHS does not depend on t.

Equilibriums: $y = y_0$, s.t. $f(y_0) = 0$

meaning: whenever $y = y_0$, $y' = 0$.

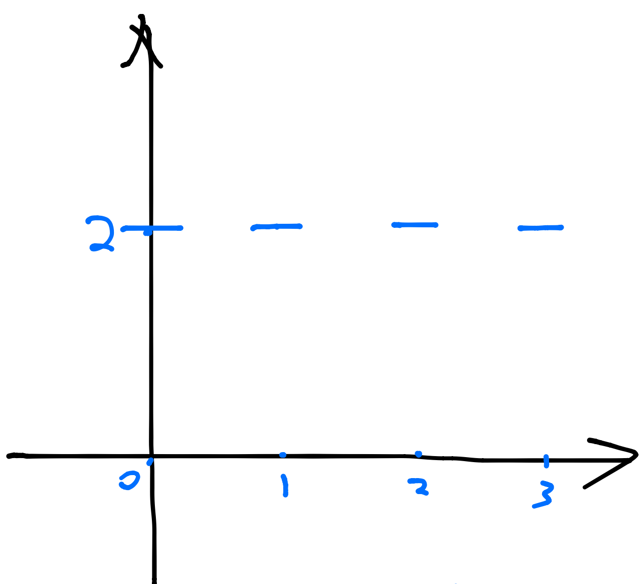
Examples: $y' = \frac{1}{2}y - 1$

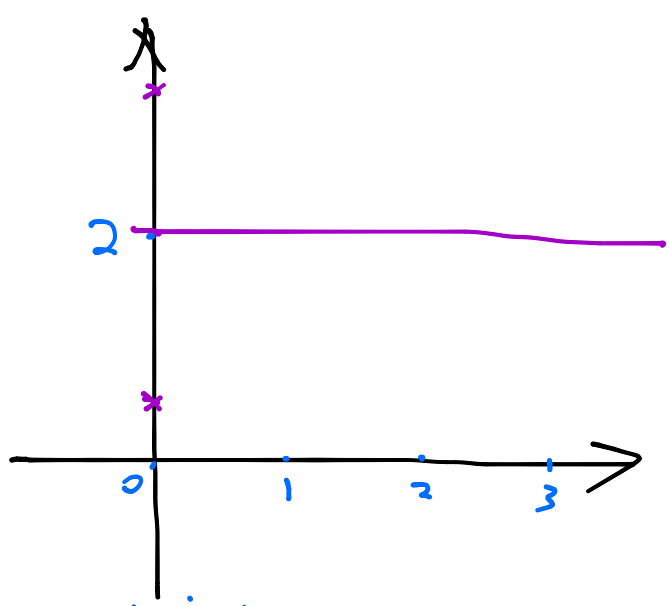
$\frac{1}{2}y_0 - 1 = 0 \Rightarrow y_0 = 2$. unique equilibrium

$y' = (y-1)y(y+1)$

$(y_0 - 1)y_0(y_0 + 1) = 0 \Rightarrow y_0 = -1, 0, 1$

3 equilibriums.





dir. field $y' = \frac{1}{2}y - 1$

trajectory

For $y' = f(y)$, if the initial value is specified such that $y(t_0)$ is an equilibrium y_0, then the solution to the IVP

$$y(t) = y_0$$ constantly

— Equilibrium solution.

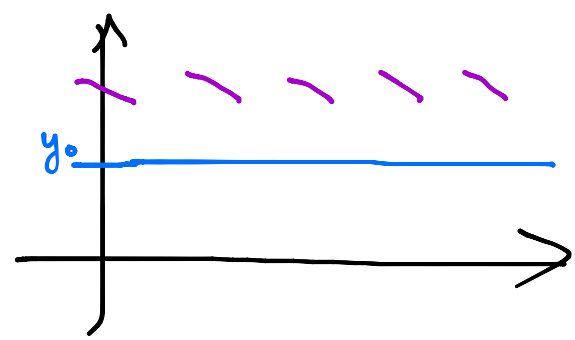
Stability: $y' = f(y)$

Let $y = y_0$ be an equilibrium solution.

$y = y_0$ is stable from above

if for $y > y_0$ not far from y_0

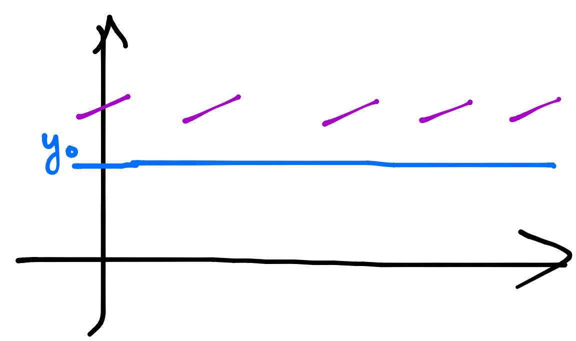
$f(y) < 0$



$y = y_0$ is unstable from above

if for $y > y_0$ not far from y_0

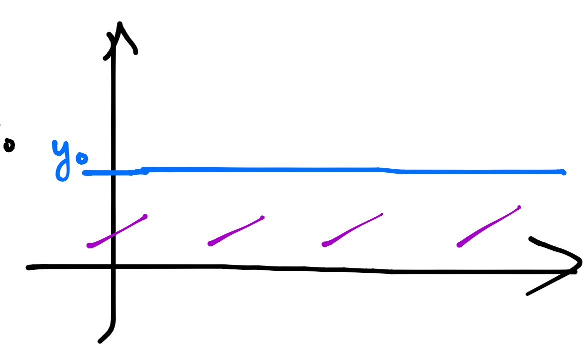
$f(y) > 0$



$y = y_0$ is stable from below

if for $y < y_0$ not far away from y_0

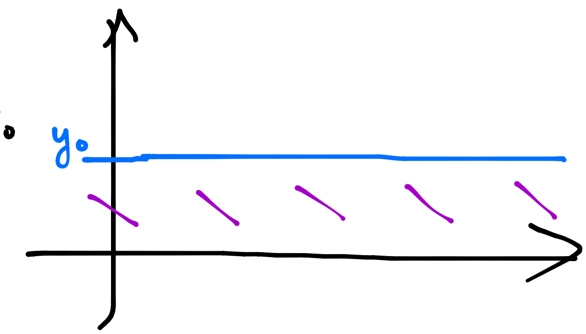
$f(y) > 0$



$y = y_0$ is unstable from below

if for $y < y_0$ not far away from y_0

$f(y) < 0$

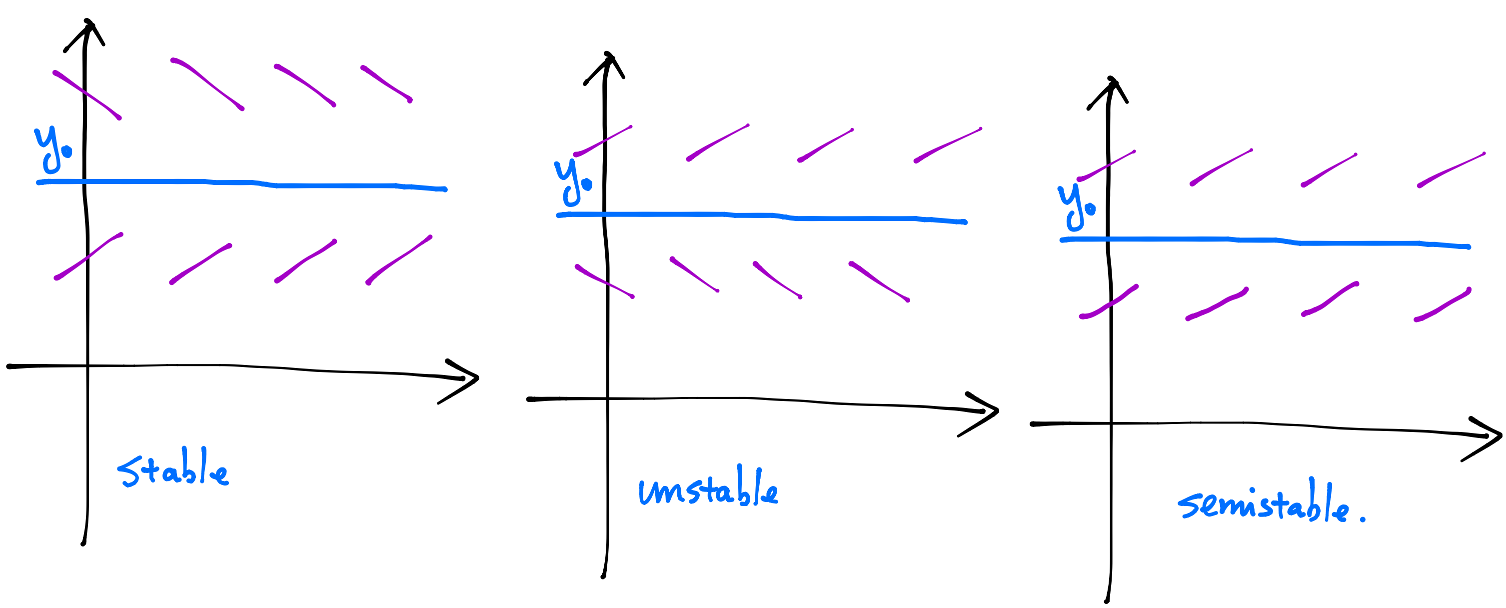


An equilibrium $y = y_0$ is called

① Stable if it's both stable from above and stable from below

② unstable if it's both unstable from above and unstable from below

③ semistable if it's stable from one side and unstable from the other side



Qualitative methods:

① Find all the equilibriums

② Find stability of all equilibriums

Example: Pollution Model:

$$\frac{dP}{dt} = \gamma - \frac{P}{V}W$$

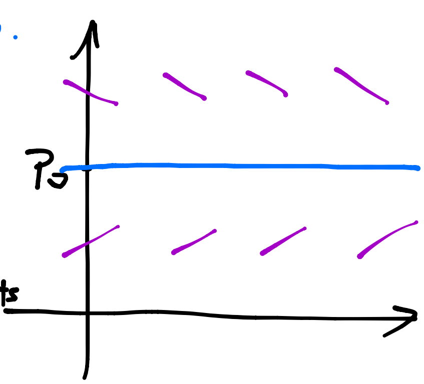
autonomous, with $f(P) = \gamma - \frac{P}{V}W$

Equilibrium: $\gamma - \frac{P_0}{V}W = 0 \Rightarrow P_0 = \frac{\gamma V}{W}$

This equilibrium is the mass of pollutant as $t \to \infty$.

b/c if $P > P_0$, $\gamma - \frac{P_0}{V}W = 0 \Rightarrow \gamma - \frac{P}{V}W < 0$, i.e. $\frac{dP}{dt} < 0$

$P < P_0$, $\gamma - \frac{P}{V}W > 0$, $\frac{dP}{dt} > 0$.



By controling the emission of pollutants, it is possible to control the total pollutants in the lake to stay at a level that water remains drinkable.

$V = 10^6 \, m^3$. $W = 2 \times 10^4 \, m^3/d$. find γ s.t. $P_0 = \frac{\gamma V}{W} < 2$

$$\gamma < \frac{2W}{V} = \frac{2 \times 2 \times 10^4}{10^6} = 0.04 \ kg/d.$$

Example: Falling object from great height

subj. to gravity $= mg$

air resistance $= kv^2$

Newton's second law: $m\dfrac{dv}{dt} = mg - kv^2$ autonomous

$$v(0) = 0$$

Qualitative method gives terminal velocity:

$$mg - kv_\infty^2 = 0 \Rightarrow v_\infty = \sqrt{\frac{mg}{k}}$$

A man and his parachute weight 192 lb.

Safe landing velocity 16 ft/sec

Air resistance = 1/2 lb/ft² of the area of the parachute

when it's moving at 20 ft/sec.

Find minimal area of the parachute s.t. terminal velocity is within the safe landing velocity.

$$mg = 192, \quad 16 = \sqrt{\frac{mg}{k}} \Rightarrow k = \frac{mg}{16^2} = \frac{192}{16\times16} = \frac{3}{4}$$

$$\text{Air resistance} = 20^2 \times \frac{3}{4} = \frac{1}{2}\,\text{Area}$$

$$\Rightarrow \text{Area} = \frac{20^2 \times 3 \times 2}{4} = 600 \quad (ft^2)$$

More examples:

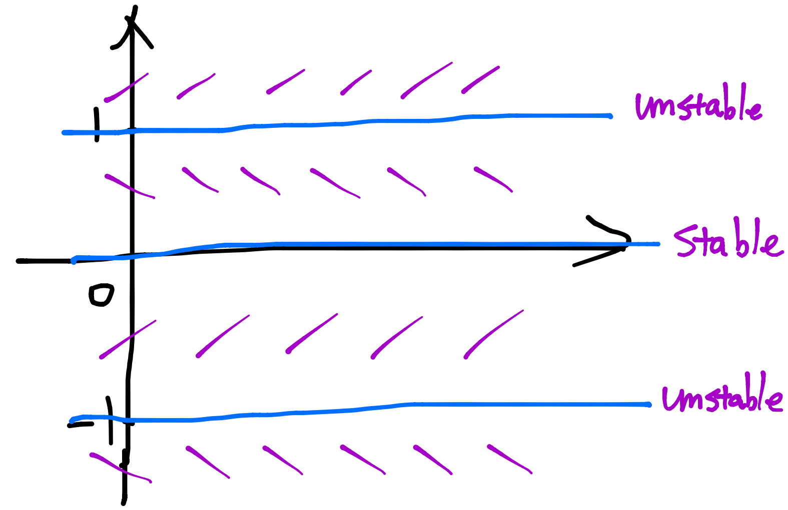
① $y' = y(y-1)(y+1)$ Equilibriums: $y_0 = -1, 0, 1$

$y > 1$. $y' > 0$.

$0 < y < 1$ $y' < 0$

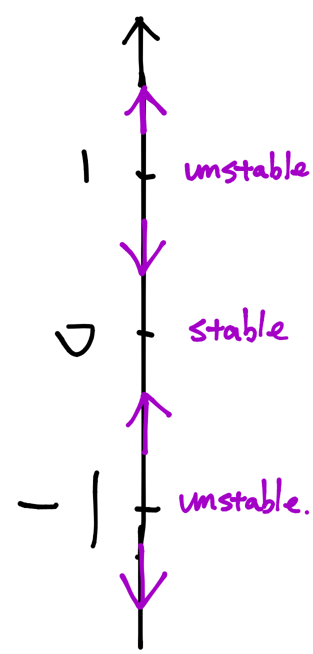
$-1 < y < 0$ $y' > 0$

$y < -1$ $y' < 0$



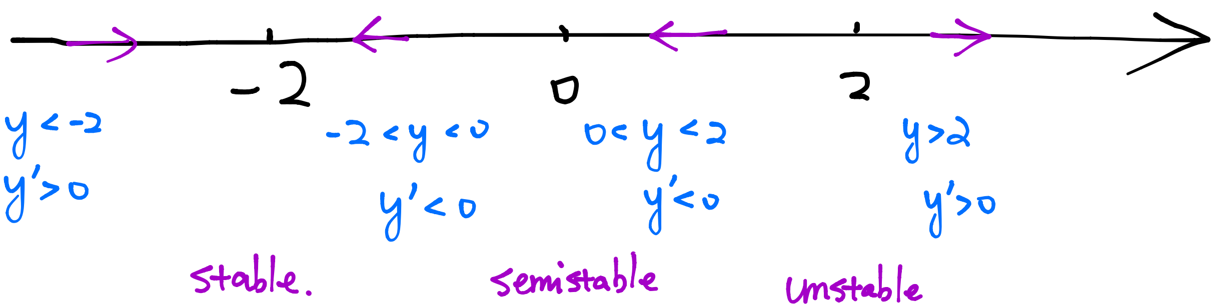
Instead of drawing the full dir. field, we can compress the two dimensional plane into one single line, called **Phase Line**.

It can be drawn either horizontally or vertically.

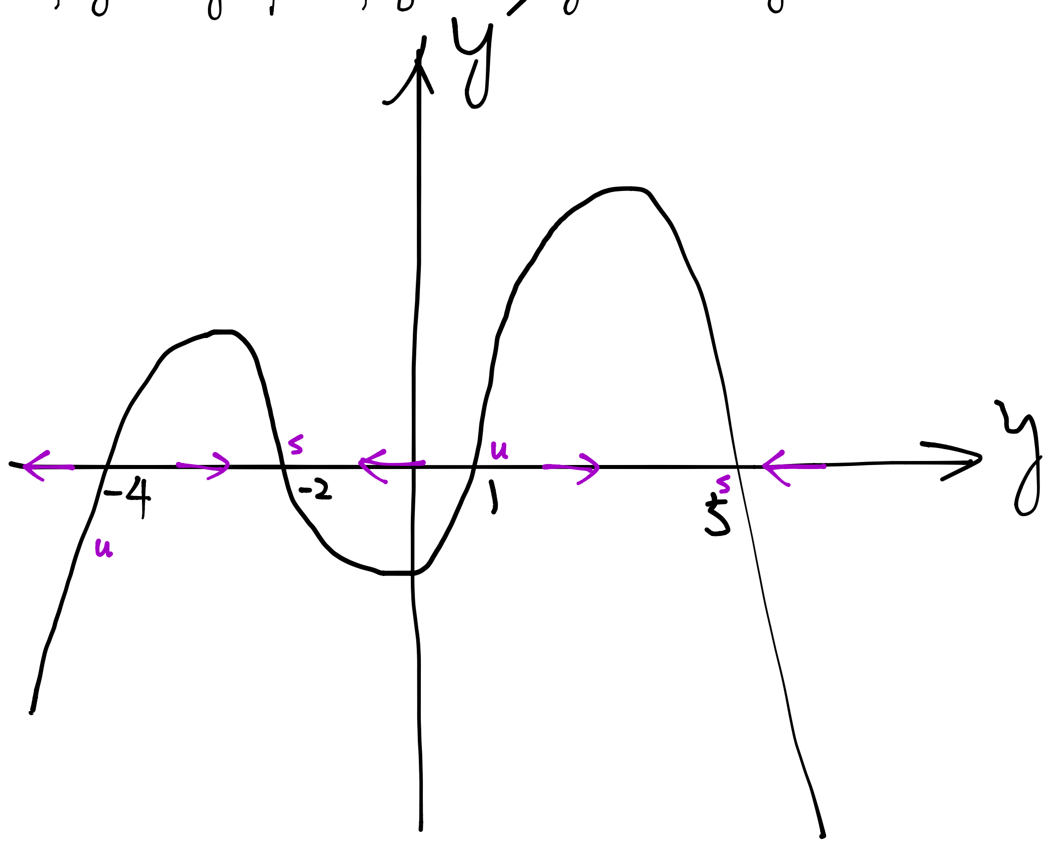


② $y' = y^2(y^2 - 4) = y^2(y-2)(y+2)$

Equilibriums: $y = -2, 0, 2$



③ $y' = f(y)$ graph $f(y)$ v.s. y will be given.

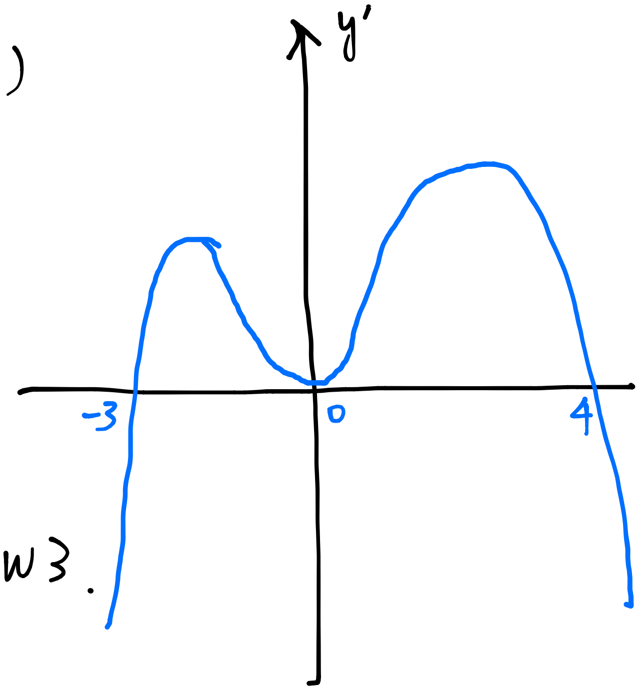


Attendance Quiz: Given the graph of $f(y)$ versus y

Find Equilibriums of $y' = f(y)$

Draw the phase line

Determine Stability.



Today's HW: skip #2

include #4 in HW3.

Lecture Notes of Differential Equation

Lecture	Page

LECTURE NOTES OF DIFFERENTIAL EQUATION

Lecture	Page

Lecture Notes of Differential Equation

Lecture	Page